RECURRENT AND NON-WANDERING PROPERTIES FOR DECOMPOSITIONS

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Abstract. In this paper, we define the recurrence and "non-wandering" for decompositions such that pointwise almost periodicity implies recurrence and that recurrence implies "non-wandering" as usual dynamical systems. Let $\mathcal{F}$ be a codimension one non-wandering C^2 foliation on a closed 3-manifold M. When there are no leaves of $\mathcal{F}$ whose ends are uncountable, we show that $\mathcal{F}$ is R-closed if and only if $\mathcal{F}$ either has no compact leaves or has no locally dense leaves. On the other hand, there are codimension one smooth foliations without compact leaves which are not R-closed but recurrent. In addition, each aperiodic non-minimal non-wandering homeomorphism isotopic to identity on a compact orientable surface is a pseudo irrational rotation.

1. Introduction and Preliminaries

In 1927, Birkhoff introduced the concepts of non-wandering points and recurrent points [Bi]. Using these concepts, we can describe and capture sustained or stationary dynamical behaviours and conservative dynamics. In this paper, we define the recurrence and "non-wandering" for decompositions such that pointwise almost periodicity implies recurrence and that recurrence implies "non-wandering" as usual dynamical systems. Using these notions, we study codimension one non-wandering foliations $\mathcal{F}$ on closed 3-manifolds M. When there are no leaves of $\mathcal{F}$ whose ends are uncountable, we show that $\mathcal{F}$ is R-closed if and only if $\mathcal{F}$ either has no compact leaves or has no locally dense leaves. On the other hand, there are codimension one smooth foliations without compact leaves which are not R-closed but recurrent. In addition, we show that each aperiodic non-minimal non-wandering homeomorphism isotopic to identity on a compact orientable surface is a pseudo irrational rotation.

By a decomposition, we mean a family $\mathcal{F}$ of pairwise disjoint nonempty subsets of a set X such that $X = \sqcup \mathcal{F}$. For $L \in \mathcal{F}$, we call that L is proper if $\overline{L} - L$ is closed. Denote by Cl (resp. P) the set of closed elements (resp. the set of elements which are not closed but proper). A decomposition $\mathcal{F}$ is recurrent (resp. non-wandering) if $P = \emptyset$ (resp. $\mathrm{int}P = \emptyset$). For any $x \in X$, denote by L_x the element of $\mathcal{F}$ containing x. Recall that $\mathcal{F}$ is pointwise almost periodic if the set of all closures of elements of $\mathcal{F}$ also is a decomposition. $\mathcal{F}$ is R-closed if and only if $R := \{(x, y) \mid y \in \overline{L_x}\}$ is closed.

Lemma 1.1. *Let $\mathcal{F}$ be a decomposition on X. If $\mathcal{F}$ is pointwise almost periodic, then $\mathcal{F}$ is recurrent. If $\mathcal{F}$ is recurrent, then $\mathcal{F}$ is non-wandering.*

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Proof. By definition, recurrence implies non-wandering property. Suppose that $\mathcal{F}$ is pointwise almost periodic. Fix any non-closed element $L \in \mathcal{F}$. Since L is not closed, there is an element $x \in \overline{L} - L$. Since $\mathcal{F}$ is pointwise almost periodic, we obtain $\overline{L_x} = \overline{L}$. Then $\overline{L_x} = \overline{L} \supsetneq \overline{L} - L = \overline{L_x} - L \supsetneq L_x$. Thus $\overline{L} = \overline{\overline{L} - L}$ and so $\overline{L} - L$ is not closed. This shows that L is not proper. Therefore $P = \emptyset$. □

2. CODIMENSION ONE FOLIATIONS

For a codimension one foliation $\mathcal{F}$ on a manifold M, denote by LD the union of locally dense leaves.

Lemma 2.1. *Let $\mathcal{F}$ be a codimension one foliation on a closed manifold. Then $\mathcal{F}$ is non-wandering if and only if $\overline{LD} \supseteq M - Cl$.*

Proof. Suppose that $\overline{LD} \supseteq M - Cl$. Obviously, $\mathrm{int}P = \emptyset$. Conversely, suppose that $\mathcal{F}$ is non-wandering. By definitions, we have $M = Cl \sqcup P \sqcup LD \sqcup E$. By Theorem 4.1.3.V [HH], we have that Cl is closed. By [Sal], there are at most finitely many local exceptional minimal sets and so E is nowhere dense. Since $\overline{E}$ has no interior, we obtain that $M - \overline{E} \subseteq Cl \sqcup P \sqcup LD$ is open dense in M. Since P has no interior, we have that $\overline{Cl \sqcup LD} \supseteq M - \overline{E}$ and so $\overline{Cl \sqcup LD} \supseteq M$. Since Cl is closed, we obtain $\overline{LD} \supseteq M - Cl$. □

We consider surface cases.

Corollary 2.2. *Let $\mathcal{F}$ be a continuous codimension one non-wandering foliation on a closed surface. Then $\mathcal{F}$ is compact or minimal. In particular, $\mathcal{F}$ is R-closed.*

For the rest of this section, we consider the case on closed 3-manifolds.

Lemma 2.3. *Let L be a leaf of a codimension one non-wandering C^2 foliation $\mathcal{F}$ on a closed 3-manifold. If L has nonempty countable ends, then L is locally dense or has non-trivial holonomy.*

Proof. By Duminy theorem for ends [CtC2], the leaf L is not exceptional. Suppose that L is proper and has trivial holonomy. By Theorem 2. [CtC], there is an open saturated neighbourhood U fibered over $\mathbb{R}$ by $\mathcal{F}|_U$. This implies that $\mathrm{int}P \neq \emptyset$ and so $\mathcal{F}$ is not non-wandering. □

The above C^2-condition is necessary. Indeed, the double suspension of the Denjoy C^1-diffeomorphism and the identity $\mathrm{id}_{\mathbb{S}^1}$ is a C^1 codimension one foliation without holonomy on $\mathbb{T}^3$ which consists of exceptional leaves with two ends and other proper leaves with one end. First we consider the case each leaves has countably many ends.

Theorem 2.4. *Let $\mathcal{F}$ be a codimension one non-wandering C^2 foliation on a closed 3-manifold M. Suppose there are no leaves of $\mathcal{F}$ whose ends are uncountable. Then $\mathcal{F}$ is R-closed if and only if $\mathcal{F}$ either has no compact leaves or has no locally dense leaves.*

Proof. Suppose that $\mathcal{F}$ is R-closed. By Theorem 5.2 [Y], we have that $\mathcal{F}$ is minimal or compact and so $\mathcal{F}$ either has no compact leaves or has no locally dense leaves. Conversely, suppose that $\mathcal{F}$ either has no compact leaves or has no locally dense leaves. By the Duminy theorem, there are no exceptional leaves. First, consider the case that there are no compact leaves. Then the non-existence of exceptional leaves

implies that the minimal set is the whole manifold M. Therefore $\mathcal{F}$ is minimal and so R-closed. Thus we may assume that there are compact leaves. Then there are no locally dense leaves. Assume that $\mathcal{F}$ is not compact. Since the union of compact leaves are closed, the union of non-compact leaves are open. Since the union of leaves without holonomy is residual (cf. [EMT]), there is a non-compact leaf L without holonomy. By Lemma 2.3, we have that L is locally dense. This contradicts to the non-existence of locally dense leaves. Thus $\mathcal{F}$ is compact and so $\mathbb{R}$-closed. $\square$

The following statement shows that the above countable condition is necessary and that pointwise almost periodicity does not correspond to recurrence.

Proposition 2.5. *There is a smooth codimension one foliation $\mathcal{F}$ on a closed 3-manifold $\Sigma \times \mathbb{S}^1$ which is not pointwise almost periodic but recurrent such that $\mathcal{F}$ consists of exceptional leaves and locally dense leaves, where Σ is the genus four closed orientable surface.*

Proof. Let G be the group generated by a circle diffeomorphisms f, g in [Sac] with a unique Cantor minimal set $\mathcal{M}$ and $f_1, f_2 : (1/3, 2/3) \to (1/3, 2/3)$ smooth diffeomorphisms such that each orbit of the group generated by f_1, f_2 is dense. Note that $(1/3, 2/3)$ is a connected component of $\mathbb{S}^1 - \mathcal{M}$. We can choose f_1, f_2 such that the extensions of f_i are circle smooth diffeomorphism $F_i : \mathbb{S}^1 \to \mathbb{S}^1$ whose supports are $(1/3, 2/3)$. Consider the product foliation $\{\Sigma \times \{x\} \mid x \in \mathbb{S}^1\}$ and four disjoint loops $\gamma_f, \gamma_g, \gamma_1, \gamma_2$ in Σ such that $\Sigma - \sqcup_{i=f,g,1,2}\gamma_i$ is a punctured disk. Taking holonomy maps $\mathrm{id} \times F_i : \gamma_i \times \mathbb{S}^1 \to \gamma_i \times \mathbb{S}^1$ for a circle bundle over Σ, we obtain a new foliation $\mathcal{F}$. Then $\mathcal{F}$ is a codimension one foliation such that each leaf is exceptional or locally dense. Therefore $\mathcal{F}$ is not pointwise almost periodic but recurrent. $\square$

The following statement shows that recurrence does not correspond to non-wandering property.

Proposition 2.6. *There is a smooth codimension one foliation on a closed 3-manifold which is not recurrent but non-wandering.*

Proof. Consider the exceptional minimal set C of $\mathbb{S}^1$ for a group G in Example 4.1.6 [CdC]. Note that the suspended foliation consists of exceptional leaves and proper leaves. We modify this foliation into a foliation with one proper leaf and other exceptional or locally dense leaves. Indeed, fix a point $p \in \mathbb{S}^1 - C$. Define smooth diffeomorphisms $f_1, f_2 : \mathbb{S}^1 \to \mathbb{S}^1$ such that the set of fixed points for F is $G(p) \sqcup C$, where F is the group generated by f_1, f_2. We also require the following conditions. For each connected component I of $\mathbb{S}^1 - C$ which does not intersect $G(p)$, each F-orbit of a point of I is dense in I. For each connected component I' of $\mathbb{S}^1 - C$ which intersects $G(p)$, let I'_i be connected components of $I' \setminus \{g(p) \mid g \in G\}$. Then each F-orbit of a point of I'_i is dense in I'_i. As above, we can construct a codimension one foliation on $\Sigma \times \mathbb{S}^1$ which consists of one proper leaf and other leaves which are either exceptional or locally dense. Hence $\mathcal{F}$ is not recurrent but non-wandering. $\square$

Moreover we state some properties of non-wandering codimension one continuous foliations. Recall that $\mathcal{F}$ is π_1-injective if each inclusion of a leaf L of $\mathcal{F}$ induces an injective map $\pi_1(L) \to \pi_1(M)$ for some base point in L.

Lemma 2.7. *Let $\mathcal{F}$ be a codimension one non-wandering foliation on a closed 3-manifold M. Then $\mathcal{F}$ is π_1-injective. Moreover if $\pi_2(M)$ is trivial, then the universal cover of M is homeomorphic to $\mathbb{R}^3$.*

Proof. By non-wandering property, there are no Reeb components. By the C^0 Novikov Compact Leaf Theorem [So], there are no vanishing cycles. By Th3.4.VIII [HH], we have that $\mathcal{F}$ is π_1-injective. Suppose that $\pi_2(M)$ is trivial. By Corollary 2.4 [P], the universal cover of M is homeomorphic to $\mathbb{R}^3$. □

This implies the following corollaries.

Corollary 2.8. *There are no codimension one non-wandering foliation on homology 3-spheres.*

By Remark 1.3. [YT], we obtain a following corollary.

Corollary 2.9. *Each non-wandering codimension one foliation by cylindrical leaves on a closed 3-manifold is without holonomy.*

3. Codimension two and "without compact elements" cases

Finally we consider the case without compact elements (i.e. aperiodic) and show that nontrivial aperiodic non-wandering surface homeomorphisms isotopic to identity must be pseudo irrational rotations. By a continuum we mean a compact connected metrizable space which is not a singleton. A continuum $A \subset X$ is said to be annular if it has a neighbourhood $U \subset X$ homeomorphic to an open annulus such that $U - A$ has exactly two components each of which is homeomorphic to an annulus. We call any such U an annular neighbourhood of A. We say a subset $C \subset X$ is a circloid if it is an annular continuum and does not contain any strictly smaller annular continuum as a subset. A homeomorphism is aperiodic if it has no periodic orbits. For a homeomorphism f on a torus or an annulus, let F the lift of f on the universal cover $\widetilde{M} \subseteq \mathbb{R}^2$. Recall that the rotation set $\rho(F) \subset \mathbb{R}^2$ is defined as follows: $\rho \in \rho(F)$ if there are a sequence $(p_i) \subset \widetilde{M}$ and an increasing sequence $(n_i) \subseteq \mathbb{Z}$ such thah $\rho = \lim_{i\to\infty} \frac{F^{n_i}(p_i)-p_i}{n_i}$. The homeomorphism f is pseudo irrational rotation if $\rho(F)$ is a single point set and f has no periodic points. Now we state the precise statement.

Proposition 3.1. *Let M be a compact orientable surface and f an aperiodic non-minimal non-wandering homeomorphism on M isotopic to identity. Then f is a pseudo irrational rotation and M is either toral or annular. Moreover f is R-closed and the orbit class space is homeomorphic to either $\mathbb{S}^1$ or $[0,1]$.*

Proof. Since f is isotopic to identity, the Lefschetz-Hopf theorem implies that the Euler number of M is zero and so that M is toral or annular. By taking a doubling of M if M is annular, we may assume that M is toral. By Theorem 1.1.[K], there is a minimal set C which is annular. We show that C is a circloid. Indeed, let U be a sufficiently small annular neighbourhood of C such that $U - C$ is a disjoint union of two open annuli A_1, A_2. Since C is f-invariant and minimal, we have that $C = \partial A_1 \cap \partial A_2$. Suppose that there is an annular continuum $C' \subsetneq C$. Then there is an annular neighbourhood U' of C' such that $U' \subset U$. Embedding U into $\mathbb{S}^2$, we may assume that U is a subset of $\mathbb{S}^2$. Then $\mathbb{S}^2 - C$ is a disjoint union of two open disks D_1, D_2 and $\mathbb{S}^2 - C'$ is a disjoint union of two open disks D_1', D_2'. Since $\mathbb{S}^2 - C' \supsetneq \mathbb{S}^2 - C$, we have $D_1 \sqcup D_2 \subsetneq D_1' \sqcup D_2'$. Since $D_1 \sqcup D_2 \sqcup \{x\}$ for any

element $x \in C$ is connected, we obtain $D'_1 \sqcup D'_2$ is connected. This contradicts to disconnectivity. Thus C is a circloid. We show that f is a pseudo irrational rotation. Indeed, let $\hat{A}$ be the Carathéodory compactification of $A := M - C$ and $\hat{f}$ the homeomorphism of $\hat{A}$ constructed in [MN]. By Theorem 1.1.[MN], the rotation numbers of $\partial\hat{A}$ are irrational. Identify $\hat{A}$ with $S^1 \times [0,1]$. Since f is non-wandering, we have that A is contained in the chain recurrent set. Then $\hat{A} = \partial\hat{A} \sqcup A$ is also contained in the chain recurrent set. By Theorem [Sw], the rotation number function $\rho = \rho_{\hat{f}} : \hat{A} \to \mathbb{R}$ is well defined and continuous. Suppose that ρ is not constant. Then fix a rotation number $p/q \in \rho(A) \cap \mathbb{Q}$, which is different from $\rho(\partial\hat{A})$. By Corollary 2.4.[F], A has a periodic point $z \in A$ with $f^q(z) = z$. This contradicts to the aperiodicity. Thus f is an irrational pseudo rotation. By the Brouwer's non-wandering Theorem [Br], each circloid is essential and homologous to each other. Since f is non-wandering, by Theorem 1.1.[K], each orbit closure is a circloid and so the orbits class space $M/\hat{f}$ is homeomorphic to $\mathbb{S}^1$. Since $M/\hat{f}$ is Hausdorff, the diagonal set $\{(p,p) \mid p \in M/\hat{f}\}$ is closed and so the inverse image $R = \{(x,y) \mid y \in \overline{O_f(x)}\}$ is closed. □

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